Note on Electrodynamics

This note is based on the textbook Classical Electrodynamics 3rd edition (John David Jackson) as well as 电动力学简明教程 (俞允强).

Vector Calculus

Scalar Triple Product

$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right)$

Vector Triple Product

BAC-CAB rule: $\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B} \times \left(\vec{A} \cdot \vec{C}\right) - \vec{C} \times \left(\vec{A} \cdot \vec{B}\right)$

Product Rules

$\nabla(fg) = f\nabla g + g\nabla f$

$\nabla\left(\vec{A} \cdot \vec{B}\right) = \vec{A} \times \left(\nabla \times \vec{B}\right) + \vec{B} \times \left(\nabla \times \vec{A}\right) + \left(\vec{A} \cdot \nabla\right)\vec{B} + \left(\vec{B} \cdot \nabla\right)\vec{A}$

$\nabla \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\nabla \times \vec{A}\right) - \vec{A} \cdot \left(\nabla \times \vec{B}\right)$

$\nabla \times \left(\vec{A} \times \vec{B}\right) = \left(\vec{B} \cdot \nabla\right)\vec{A} - \left(\vec{A} \cdot \nabla\right)\vec{B} + \vec{A}\left(\nabla \cdot \vec{B}\right) - \vec{B}\left(\nabla \cdot \vec{A}\right)$

$\nabla \cdot \left(f\vec{A}\right) = f\left(\nabla \cdot \vec{A}\right) + \vec{A} \cdot (\nabla f)$

$\nabla \times \left(f\vec{A}\right) = f\left(\nabla \times \vec{A}\right) - \vec{A} \times (\nabla f)$

Second Derivatives

$\nabla \cdot (\nabla T) = (\nabla \cdot \nabla)T = \nabla^2 T$

$\nabla \times (\nabla T) = 0$

$\nabla(\nabla \cdot \vec{v})$

$\nabla \cdot (\nabla \times \vec{v}) = 0$

$\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$

Coordinates

Cylindrical Coordinates

Theorem 1 (**Gradient in Cylindrical Coordinates**):

$$\nabla = \hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{e}_z \frac{\partial}{\partial z} \tag{1}$$

Theorem 2 (**Laplacian in Cylindrical Coordinates**):

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \tag{2}$$

Special Functions

Definition 1 (**Complete elliptic integral of the first kind**):

$$K(k) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{3}$$

Definition 2 (**Complete elliptic integral of the second kind**):

$$E(k) = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\,\mathrm{d}x \tag{4}$$

1. Introduction to Electrostatics

Theorem 3 (**Green's first identity**):

$$\int_V \left(\varphi\nabla^2\psi + \nabla\varphi \cdot \nabla\psi\right) \mathrm{d}^3x = \oint_S \varphi \frac{\partial \psi}{\partial n}\,\mathrm{d}a \tag{5}$$

proof: Substitute $\vec{A}$ in divergence theorem with $\varphi\nabla\psi$. ∎

Theorem 4 (**Green's second identity**):

$$\int_V \left(\varphi\nabla^2\psi - \psi\nabla^2\varphi\right) \mathrm{d}^3x = \oint_S \left(\varphi\frac{\partial \psi}{\partial n} - \psi\frac{\partial \varphi}{\partial n}\right) \mathrm{d}a \tag{6}$$

proof: interchange φ and ψ in Green's first identity and then substract. ∎

1.1. Poisson and Laplace Equations

The behavior of an electrostatics field is described by

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{7}$$

$$\nabla \times \vec{E} = 0 \tag{8}$$

Definition 3 (**Poisson equation**): The electric potential Φ satisfies the equation

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{9}$$

Definition 4 (**Laplacian equation**): In regions of space lacking charge, the Poisson equation becomes

$$\nabla^2\Phi = 0 \quad (10)$$

1.2. Solution of Boundary-Value Problem with Green Function

Theorem 5 (**Gauss's theorem**):

$$\oint \vec{E}\cdot \mathrm{d}\vec{a} = \frac{q}{4\pi\varepsilon_0}\int \mathrm{d}\Omega \quad (11)$$

Definition 5 (**Green function**): A function

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}') \quad (12)$$

must satisfy the condition that:

$$\nabla'^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') \quad (13)$$

And with F satisfying the Laplace equation inside the volume V

Theorem 6 (**general solution for Poisson function**):

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x},\vec{x}')\,\mathrm{d}^3x' + \frac{1}{4\pi}\oint_S\left(G(\vec{x},\vec{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right)\mathrm{d}a' \quad (14)$$

proof: Plug $G(\vec{x},\vec{x}')$ and Φ into Eq. 6 ■

Theorem 7: solution of Poisson equation with Dirichlet or Neumann boundary conditions is unique

proof: Let $U = \Phi_1 - \Phi_2$ and use Theorem 3. ■

Definition 6 (**Dirichlet boundary conditions**):

$$G_D(\vec{x},\vec{x}') = 0 \text{ for } \vec{x} \text{ on S} \quad (15)$$

Theorem 8 (**Solution to Dirichlet boundary conditions**):

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')G_D(\vec{x},\vec{x}')\,\mathrm{d}^3x - \frac{1}{4\pi}\oint \Phi(\vec{x}')\frac{\partial G_D}{\partial n'}\,\mathrm{d}a' \quad (16)$$

Definition 7 (**Neumann boundary conditions**): This is consistent with Gauss' s theorem that

$$\frac{\partial}{\partial n'}G_N(\vec{x},\vec{x}') = -\frac{4\pi}{S} \text{ for } \vec{x} \text{ on S} \quad (17)$$

Theorem 9 (**Solution to Neumann boundary conditions**):

$$\Phi(\vec{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')G_N(\vec{x},\vec{x}')\,\mathrm{d}^3x + \frac{1}{4\pi}\oint G_N\frac{\partial\Phi}{\partial n'}\,\mathrm{d}a' \quad (18)$$

1.3. Energy and Capacitance

Theorem 10 (**Discrete total potential**):

$$W = \frac{1}{8\pi\varepsilon_0}\sum_i\sum_j \frac{q_iq_j}{|\vec{x}_i-\vec{x}_j|} \quad (19)$$

Theorem 11 (**Continuous total potential**):

$$\begin{aligned} W &= \frac{1}{8\pi\varepsilon_0}\int\int\frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}_i-\vec{x}_j|}\,\mathrm{d}^3x\,\mathrm{d}^3x' \\ &= \frac{1}{2}\int_V \rho(\vec{x})\Phi(\vec{x})\,\mathrm{d}^3x \\ &= -\frac{\varepsilon_0}{2}\int \Phi\nabla^2\Phi\,\mathrm{d}^3x \end{aligned} \quad (20)$$

With self-energy contributions

$$\begin{aligned} W &= \frac{\varepsilon_0}{2}\int|\boldsymbol{\nabla}\Phi|^2\,\mathrm{d}^3x \\ &= \frac{\varepsilon_0}{2}\int|\vec{E}|^2\,\mathrm{d}^3x \end{aligned} \quad (21)$$

Definition 8 (**Energy density**): With self-energy contributions

$$w = \frac{\varepsilon_0}{2}|\vec{E}|^2 \quad (22)$$

2. Boundary-Value Problems in Electrostatics

2.1. Method of Images

What are image charges A small number of charges

- suitably placed
- appropriately charged
- external to the region of interest

- simulating the required boundary conditions

Zero potential plane conductor condition

$q' = -q$ and $x' = -x$

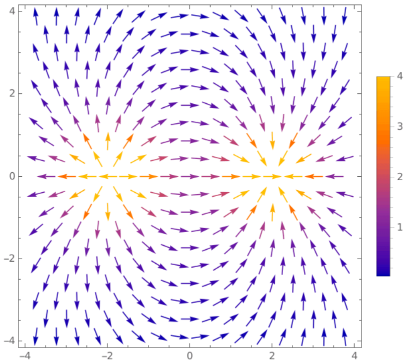

Figure 1: Electric filed of q away from an infinite plane conductor

hollow grounded sphere conductor

2.2. Laplace Equation in Rectangular Coordinates

2.3. Fields in Two-Dimensional Corners

2.4. Expansion in Spherical Coordinates

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \quad (23)$$

3. Multipoles and Dielectrics

3.1. Multipole Expansion

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \quad (24)$$

Definition 9 (**traceless quadrupole moment**):

$$Q_{ij} = \int \left(3x_i'x_j' - r'^2\delta_{ij}\right)\rho(\vec{x}')\, \mathrm{d}^3x' \quad (25)$$

$$\vec{p} = \int \vec{x}'\rho(\vec{x}')\, \mathrm{d}^3x' \quad (26)$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \ldots\right) \quad (27)$$

4. Relativistic Electromagnetics

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \partial^\mu = \frac{\partial}{\partial x_\mu} \quad (28)$$